The Unit Circle

Things you should already know

Fact (Trigonometric Ratios) — For a right-angled triangle with angle θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Fact (SOHCAHTOA) — **S**in = **O**pp / **H**yp, **C**os = **A**dj / **H**yp, **T**an = **O**pp / **A**dj

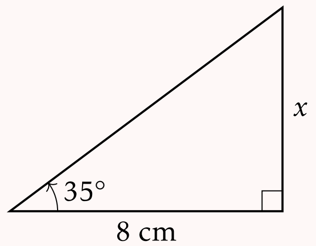
These definitions work for angles between 0° and 90° in a right-angled triangle.

Fact (Calculator Functions) — Your calculator has sin, cos, and tan buttons — these take an angle and return the ratio.

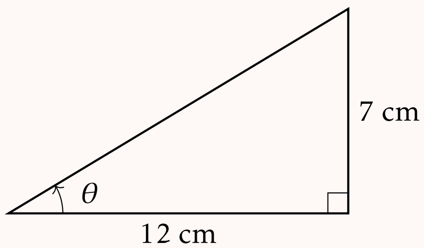
It also has $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ (sometimes written arcsin, arccos, arctan) — these take a ratio and return the angle.

Example

Find the length x and the angle θ in the triangles below.



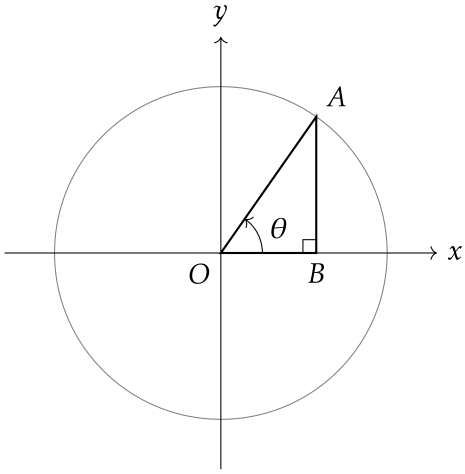
(not drawn accurately)



Triangle 1: $\tan 35° = \frac{x}{8}$, *so* $x = 8\tan 35° = 5.60$ *cm (3 s.f.)*

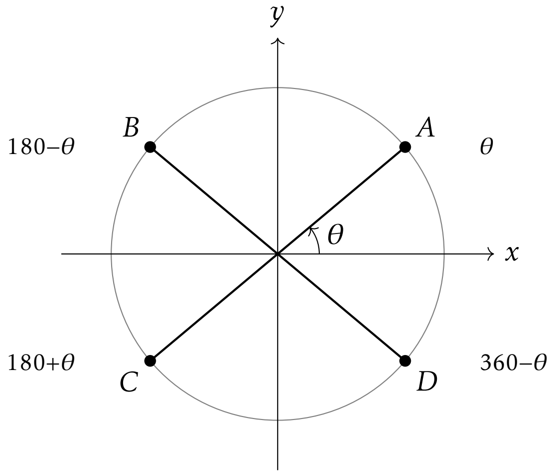
Triangle 2: $\tan\theta = \frac{7}{12}$, *so* $\theta = \tan^{-1}\left(\frac{7}{12}\right) = 30.3°$ *(1 d.p.)*

The Unit Circle



The coordinates of A are $(\cos\theta, \sin\theta)$*. The vertical side is* $\sin\theta$*, the horizontal side is* $\cos\theta$*, and the radius is* 1*.*
This definition works for ***all*** *angles — not just* $0°$ *to* $90°$*.*
$\tan\theta$ *is* ***defined*** *as* $\dfrac{\sin\theta}{\cos\theta}$.

The Four Quadrants



Example

In which quadrants is each trig function positive?

Quadrant	*Angle range*	*Positive functions*
1st	$0°–90°$	sin, cos, tan
2nd	$90°–180°$	sin *only*
3rd	$180°–270°$	tan *only*
4th	$270°–360°$	cos *only*

Remember: ***All Students Take Coffee*** *(or* ***ASTC****) — reading anticlockwise from the 1st quadrant.*

Example

Without a calculator, determine the sign of each:

(a) $\sin 130°$ (b) $\cos 210°$ (c) $\tan 315°$

(a) $130°$ *is in the 2nd quadrant, so* $\sin 130° > 0$ *(positive)*

(b) $210°$ *is in the 3rd quadrant, so* $\cos 210° < 0$ *(negative)*

(c) $315°$ *is in the 4th quadrant, so* $\tan 315° < 0$ *(negative)*

Fact (Exact Values) —

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Example

Without a calculator, calculate the value of each:

(a) $\sin 330°$ (b) $\cos 225°$ (c) $\tan 240°$

(a) *330° is in the 4th quadrant (sin negative), reference angle* $360° - 330° = 30°$, *so* $\sin 330° = -\sin 30° = -\frac{1}{2}$

(b) *225° is in the 3rd quadrant (cos negative), reference angle* $225° - 180° = 45°$, *so* $\cos 225° = -\cos 45° = -\frac{\sqrt{2}}{2}$

(c) *240° is in the 3rd quadrant (tan positive), reference angle* $240° - 180° = 60°$, *so* $\tan 240° = \tan 60° = \sqrt{3}$

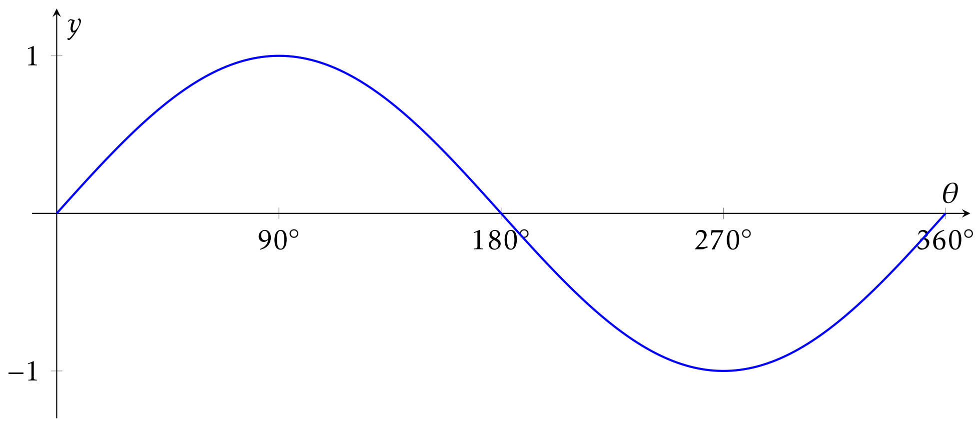
Graphs of the Trigonometric Functions

Example

Using the unit circle, fill in the table for $y = \sin\theta$:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$													

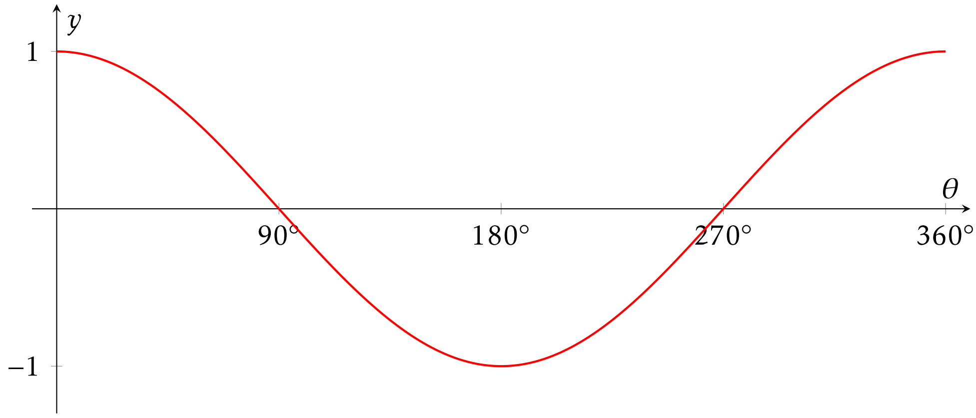
The graph of $y = \sin\theta$



Key properties of $y = \sin\theta$:

- *Maximum value:* 1 *(at* $\theta = 90°$*)*
- *Minimum value:* -1 *(at* $\theta = 270°$*)*
- *Range:* $-1 \le \sin\theta \le 1$
- *Period:* 360°
- *Symmetry:* $\sin\theta = \sin(180° - \theta)$

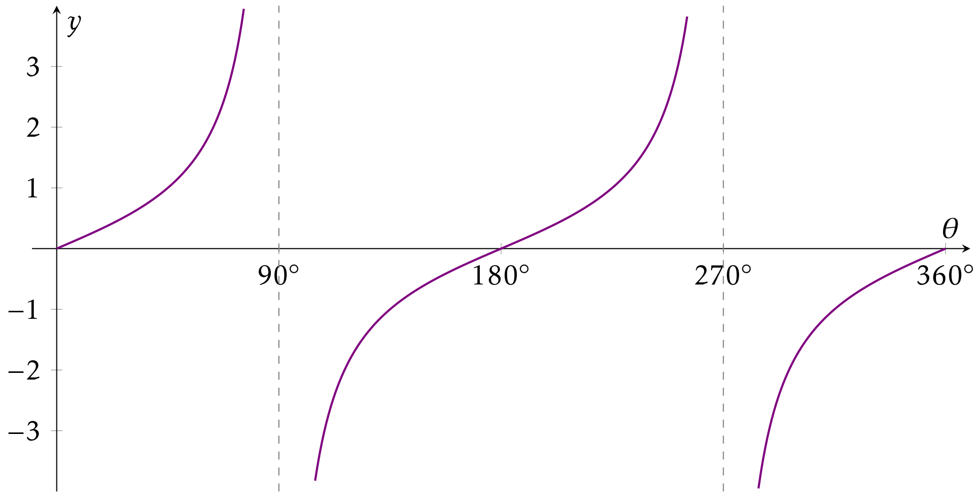
The graph of $y = \cos\theta$



Key properties of $y = \cos\theta$:

- *Maximum value:* 1 *(at* $\theta = 0°$ *and* $360°$*)*
- *Minimum value:* -1 *(at* $\theta = 180°$*)*
- *Range:* $-1 \le \cos\theta \le 1$
- *Period:* $360°$
- *Symmetry:* $\cos\theta = \cos(360° - \theta)$

The graph of $y = \tan\theta$



Key properties of $y = \tan\theta$:

- $\tan\theta$ *is* ***undefined*** *at* $\theta = 90°$ *and* $270°$ *(vertical asymptotes)*
- *Range:* $\tan\theta$ *can take* ***any value*** *(no maximum or minimum)*
- *Period:* $180°$
- *Symmetry:* $\tan\theta = \tan(180° + \theta)$

Example

You are told that $\sin 24° = 0.407$. Without a calculator, write down the values of:

(a) $\sin 156°$ (b) $\sin 204°$ (c) $\sin 336°$

Using the symmetry of the sine graph: $\sin\theta = \sin(180° - \theta)$.

(a) $\sin 156° = \sin(180° - 24°) = \sin 24° = 0.407$

(b) $\sin 204° = \sin(180° + 24°) = -\sin 24° = -0.407$

(c) $\sin 336° = \sin(360° - 24°) = -\sin 24° = -0.407$

Solving Trigonometric Equations

Example

Using a sketch of $y = \sin\theta$ for $0^\circ \le \theta \le 360^\circ$, how many solutions does $\sin\theta = 0.5$ have? What about $\sin\theta = -0.3$?

Method

Fact — To solve a trigonometric equation over $0^\circ \le \theta \le 360^\circ$:

1. Use your calculator to find the **first solution** (the "principal value").
2. Use the **symmetry of the graph** to find the **second solution**.
3. Check both solutions are in the required range.

Example

Solve $\sin\theta = 0.6$ for $0^\circ \le \theta \le 360^\circ$. Give answers to 1 d.p.

$\theta_1 = \sin^{-1}(0.6) = 36.9^\circ$
$\theta_2 = 180^\circ - 36.9^\circ = 143.1^\circ$
Solutions: $\theta = 36.9^\circ$ *or* $\theta = 143.1^\circ$

> **Example**
> Solve $\cos\theta = -0.4$ for $0° \le \theta \le 360°$. Give answers to 1 d.p.

$\cos^{-1}(-0.4) = 113.6°$ *(calculator gives the value in the 2nd quadrant)*
$\theta_1 = 113.6°$
$\theta_2 = 360° - 113.6° = 246.4°$
Solutions: $\theta = 113.6°$ *or* $\theta = 246.4°$

> **Example**
> Solve $\tan\theta = -2$ for $0° \le \theta \le 360°$. Give answers to 1 d.p.

$\tan^{-1}(-2) = -63.4°$ *(calculator gives a negative angle — not in range)*
Add $180°$*:* $\theta_1 = -63.4° + 180° = 116.6°$
$\theta_2 = 116.6° + 180° = 296.6°$
Solutions: $\theta = 116.6°$ *or* $\theta = 296.6°$

Example

Solve $\sin\theta = -0.5$ for $0° \leq \theta \leq 360°$.

$\sin^{-1}(-0.5) = -30°$ *(not in range)*
Sine is negative in the 3rd and 4th quadrants:
$\theta_1 = 180° - (-30°) = 210°$
$\theta_2 = 360° + (-30°) = 330°$
Solutions: $\theta = 210°$ *or* $\theta = 330°$

Extension: Equations of the form $\sin k\theta = c$

Fact — For equations like $\sin 2\theta = c$ or $\cos 3\theta = c$:

1. Let $u = k\theta$ and solve $\sin u = c$ in the range $0° \leq u \leq 360k°$
2. Divide all solutions by k to get θ

There are usually $2k$ solutions in any $360°$ range.

Example

Solve $\cos 2\theta = 0.5$ for $0° \leq \theta \leq 360°$.

Let $u = 2\theta$. *We need* $\cos u = 0.5$ *for* $0° \leq u \leq 720°$.
$u_1 = \cos^{-1}(0.5) = 60°$
$u_2 = 360° - 60° = 300°$
$u_3 = 60° + 360° = 420°$
$u_4 = 300° + 360° = 660°$
Dividing by 2: $\theta = 30°, 150°, 210°, 330°$

Extension: Quadratic Trigonometric Equations

Example

Solve $2\sin^2\theta - \sin\theta - 1 = 0$ for $0° \le \theta \le 360°$.

Let $x = \sin\theta$. *Then* $2x^2 - x - 1 = 0$, *so* $(2x+1)(x-1) = 0$.
$x = -\frac{1}{2}$ *or* $x = 1$. *Both are in the range* $-1 \le x \le 1$, *so both are valid.*
$\sin\theta = 1 \Rightarrow \theta = 90°$
$\sin\theta = -\frac{1}{2} \Rightarrow \theta = 210°$ *or* $\theta = 330°$
Solutions: $\theta = 90°, 210°, 330°$

Example

Solve $2\cos^2\theta - 5\cos\theta + 2 = 0$ for $0° \le \theta \le 360°$.

Let $x = \cos\theta$. *Then* $2x^2 - 5x + 2 = 0$, *so* $(2x-1)(x-2) = 0$.
$x = \frac{1}{2}$ *or* $x = 2$. *But* $-1 \le \cos\theta \le 1$, *so* $x = 2$ *is rejected.*
$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$ *or* $\theta = 300°$
Solutions: $\theta = 60°, 300°$